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A note on Landen's transformation

by

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In a private communication Balth. van der Pol once made the intriguing remark that the well-known Landen transformation of the complete elliptic integral of the first kind

$$(1) \qquad \int_0^{\frac{1}{2}\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{-\frac{1}{2}}\, d\,\theta = $$

$$= \int_0^{\frac{1}{2}\pi} (a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta)^{-\frac{1}{2}}\, d\,\theta \quad ,$$

where a_1 and b_1 are respectively the geometric and the arithmetic mean of the positive real numbers a and b, is an immediate consequence of the derivation of the spatial potential of a charged circular wire in two different ways, both well-known in potential theory.

In vain we searched his many publication for this or a similar statement. However, when discussing this problem with a few members of the "Mathematisch Centrum", notably A. van Wijngaarden, D.J. Hofsommer and R.P. van de Riet, the required result was soon rediscovered.

Prompted by the continuous interest people showed in this apparently unknown fact we prepared this little note as a posthumous tribute to the inventiveness of the renowned mathematical physicist Balth. van der Pol.

In fact, consider the three-dimensional potential u (r, z) at the point P (x, y, z) with $x^2 + y^2 = r^2$ due to a charged ring $z = o$, $r = R$. Then with the omission of an uninteresting constant factor we have

$$(2) \qquad u\,(r,\, z) = \int_{-\pi}^{\pi} (z^2 + r^2 + R^2 - 2r\,R \cos\theta)^{-\frac{1}{2}}\, d\,\theta,$$

which is known as the Poisson representation. On the other hand we may use the Helmholtz representation for the potential of a circular symmetric field

$$(3) \qquad u\,(r,\, z) = \frac{1}{\pi}\int_0^{\pi} F\,(z + i\,r \cos\theta)\, d\,\theta.$$

The integrand function F (z) is fixed by the potential at the z- axis which is of course $2\,\pi\,(z^2 + R^2)^{-\frac{1}{2}}$. This gives the equivalent integral representation

$$(4) \qquad u\,(r,\ z) = 2 \int_0^{\pi} ((z + i\ r \cos\theta)^2 + R^2)^{-\frac{1}{2}}\, d\theta.$$

The expressions (2) and (4) will now be compared at a point of the z-plane. Substitution of the values

$$(5) \qquad z = o,\ R = \tfrac{1}{2}\,(a + b),\ r = \tfrac{1}{2}\,(-a + b),$$

where a and b are positive numbers with $a < b$, into the first expression (2) gives

$$u\,(r,\ o) = 2 \int_0^{\pi} (\tfrac{1}{2}\,(a^2 + b^2) + \tfrac{1}{2}\,(a^2 - b^2) \cos\theta)^{-\frac{1}{2}}\, d\theta$$

which can be reduced to

$$(6) \qquad u\,(r,\ o) = 4 \int_0^{\frac{1}{2}\pi} (a^2 \cos^2\theta + b^2 \sin^2\theta)^{-\frac{1}{2}}\, d\theta.$$

Substitution of (5) into the second expression (4) gives

$$u\,(r,\ o) = 2 \int_0^{\pi} ((R^2 - r^2) \cos^2\theta + R^2 \sin^2\theta)^{-\frac{1}{2}}\, d\theta$$

or

$$(7) \qquad u\,(r,\ o) = 4 \int_0^{\frac{1}{2}\pi} (\ a\ b \cos^2\theta + \tfrac{1}{4}\,(a + b)^2 \sin^2\theta)\, d\theta.$$

The identity of (6) and (7) yields the Landen transformation in the form (1).

<u>Appendix 1</u>

For the benefit of the reader we collect a few well-known related facts. Both sides of (1) represent the Complete Elliptic integral of the first kind K (k) in the form

$$(8) \qquad \int_0^{\frac{1}{2}\pi} (a^2 \cos^2\theta + b^2 \sin^2\theta)^{-\frac{1}{2}}\, d\theta = b^{-1}\, K\,(\sqrt{1 - a^2/b^2}),$$

where

$$K\,(k) = \tfrac{1}{2}\,\pi\ {}_2F_1\,(\tfrac{1}{2},\ \tfrac{1}{2};\ 1;\ k^2).$$

Landen's transformation (1) can be written as the functional relation

$$K\left(\frac{1-x}{1+x}\right) = \tfrac{1}{2}(1+x)\,K\left(\sqrt{1-x^2}\right)$$

or as

$$(1+y^2)\,K(y) = K\left(\frac{4y}{(1+y)^2}\right).$$

This corresponds to a well-known transformation of the hypergeometric function which is due to Gauss. Repeated use of (1) according to the recurrence relation

$$a_{n+1} = \sqrt{a_n b_n}\,,\quad b_{n+1} = \tfrac{1}{2}(a_n + b_n)$$

with $a_o = a$ $(a < 1)$ and $b_o = 1$ yields the well-known numerical calculation of the function $K(k)$ according to

$$\lim a_n = \lim b_n = \frac{\pi}{2\,K\left(\sqrt{1-a^2}\right)}.$$

Appendix 2

The Landen transformation is usually obtained from the theory of the theta functions. The transformation (1) is a special case of a more general transformation of an incomplete elliptic integral*)

$$(9)\qquad \int_o^{\phi} (b_2 - a^2 \sin^2\theta)^{-\frac{1}{2}}\,d\theta = \tfrac{1}{2}\int_o^{\psi} (b_1^2 - a_1^2 \sin^2\theta)^{-\frac{1}{2}}\,d\theta,$$

where a, b, a_1 and b_1 $(a < b)$ have the same meaning as in (1) and where

$$\psi = \arcsin\frac{(a+b)\sin\phi}{a\sin^2\phi + b},\quad o \leq \phi \leq \tfrac{1}{2}\pi.$$

In fact for $\phi = \psi = \frac{1}{2}\pi$ the latter transformation reduces to (1) with an appropriate change in the parameters a and b. The transformation (9) can also be proved in the following elementary way. We note that (9) can be written as

*) Whittaker and Watson p 507.

$$(10) \qquad \int_0^x \{ (1 - b\,t^2)(1 - b^{-1}\,t^2) \}^{-\frac{1}{2}}\,d\,t =$$

$$= \tfrac{1}{2} \int_0^y \{ (1 - t^2)(1 - c^{-2}\,t^2) \}^{-\frac{1}{2}}\,d\,t,$$

where $y = \dfrac{2x}{1 + x^2}$ and $c = \dfrac{2\,b^{\frac{1}{2}}}{1 + b}$.

If now upon the integrand on the right-hand side the following familiar substitution is performed,

$$\sqrt{1 - t^2} = 1 - ty,$$

which serves to rationalize integrands of the form $F\,(t, \sqrt{1 - t^2})$, the required result is obtained almost at once.